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In this paper we give general criteria to be satisfied by a slowly time-dependent Hamiltonian in order to possess adiabatic invariants of m-th order. We evaluate also the degree of approximate constancy of such adiabatic invariants. We end up the paper applying our methods to the motion of a charged particle in a magnetic field, i.e. to the harmonic oscillator.

§ 1. Introduction

In plasma physics the so-called guiding center approximation treats the motion of charged particles in a varying electromagnetic field as a gyration about the guiding center which in turn moves through space. The separation of these two motions can be valid only if the change of the field in space and time is slow.

To study those motions the concept of adiabatic invariant is important. Precise formulation of the notion of adiabatic invariance has been given by Chandrasekhar[1] and Lenard.[2] Kulsrud[3] believes that there are many quantities that are adiabatic invariants of higher order, even though we only know that they are invariants to a lower order.

If the guiding center approximation is valid, the adiabatic invariant of the gyration is assumed constant. It is interesting to know the error made when this is done. Therefore we should try to study the approximate constancy of adiabatic invariants. Adiabatic invariants are strict constants of the motion if the fields are constant.

Lenard proved the adiabatic invariance of the action integral of a one-dimensional non-linear oscillator to all orders. His paper is a generalization of the work of Kulsrud who studied the same concept for the harmonic oscillator. In Lenard's paper the time-dependent Hamiltonian corresponds, for fixed time, to a periodic motion.

In this paper we will generalize the results obtained by other authors in several points. First of all, we will not require that the time-dependent hamiltonian represent instantaneously a periodic motion; the unperturbed hamiltonian, though, must correspond to a periodic motion. Secondly, we give general criteria to be satisfied by the slowly time-dependent hamiltonian in order to possess adiabatic invariants of m-th order. Thirdly, our method allows us to evaluate the degree of approximate constancy of such adiabatic invariants.

Adiabatic invariants are quantities that remain constants of the motion during an infinitely slow variation of the external parameters of the system. Their usefulness lies in the fact that they remain constant to a very good approximation even when the external parameters vary at a finite rate. In real plasma physics the external parameters always vary at a finite, though slow, rate. That is why our calculations can be applied to actual plasma problems. The method used in this paper is based on the introduction of the interaction picture[4] for classical mechanics.

We end up our paper applying the general techniques presented in the same to the motion of a charged particle in a magnetic field that varies in time slowly but at a finite rate. We evaluate the errors that are made in plasma physics when the adiabatic invariants are taken to be exact constants of the motion.

§ 2. Adiabatic invariants

We suppose that the Hamiltonian $H(t)$ of the system depends explicitly on time. As a matter of fact such a Hamiltonian, $H(t)$, varies continuously with time from an initial value, H_0, at the instant t_0 to a certain final value, H_1, at the instant t_1. We write

$$T=t_1-t_0, \qquad \tau=\frac{t-t_0}{T} \tag{1}$$

and design by $H(\tau)$ the value of the Hamiltonian $H(t)$ at the time instant $t=t_0+\tau T$. $H(\tau)$ is a continuous function of τ that we suppose given. The rate of evolution of the system from time t_0 till time t_1 depends only on the parameter T. We plan to study the case in which T is very large.

If we define the parameter $\varepsilon=1/T$, the limit $\varepsilon\to 0$ implies the indefinite decrease in the rates of change of external parameter. While the physical time t goes from t_0 to t_1, the fictitious time parameter τ changes from $\tau=0$ to $\tau=1$.

Therefore

$$H(0)=H_0, \qquad H(1)=H_1. \tag{2}$$

We have to study the evolution of the system under the action of the hamiltonian $H(\tau)$ when τ goes from $\tau=0$ to $\tau=1$. We plan to expand the corresponding evolution operator in powers of T^{-1}.

A quantity $\mathcal{L}$ is an adiabatic invariant to the m-th order, if a positive constant M can be found such that during the time interval $T(T\to\infty)$ the variation of $\mathcal{L}$ satisfies

$$|\Delta\mathcal{L}|<\frac{1}{T^m}M. \tag{3}$$

Even though we have studied elsewhere[4] the interaction picture for classical mechanics, we will do it again now for the case in which the Hamiltonian $H(t)$ depends explicitly on time in order to write the corresponding equations in terms of the fictitious time τ and of the fictitious Hamiltonian $H(\tau)$.

§ 3. Interaction picture

The dynamical time evolution of a function $A=A(q,p)$ of the sets of canonical conjugate variables, i.e. the total time evolution if that function A does not depend explicitly on time, is given by

$$\frac{dA}{dt}=\sum_i \frac{\partial H}{\partial p_i}\frac{\partial A}{\partial q_i}-\frac{\partial H}{\partial q_i}\frac{\partial A}{\partial p_i} \tag{4}$$

where $H=H(q,p,t)$ is the Hamiltonian of the system referred to the physical time t. The same equation written in terms of the new time variable τ and of the Hamiltonian $H(\tau)$ is

$$\frac{dA}{d\tau}=T[\Omega, A], \tag{5}$$

where the square parenthesis stands for the commutator between the quantities within. The operator Ω is Liouiville's differential operator given by

$$\Omega=\sum_i \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i}-\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}. \tag{6}$$

The time evolution may be integrated introducing a differential operator $S(\tau)$ written as a function of Liouiville's operator Ω. Such operator $S(\tau)$ is defined by means of

$$A(q(\tau),p(\tau))=S(\tau)\,A(q_0,p_0)\,S^{-1}(\tau) \tag{7}$$

where q_{i0} and p_{i0} are the values of the sets of canonical variables at the time origin, i.e.

$$q_{i0}=q_i(0), \qquad p_{i0}=p_i(0). \tag{8}$$

The boundary condition satisfied by $S(\tau)$ is evidently

$$S(0)=1. \tag{9}$$

If we enter with (7) into (5) we obtain the differential equation satisfied by the evolution operator

$$\frac{dS(\tau)}{d\tau}=TS(\tau)\Omega(q_0,p_0,\tau). \tag{10}$$

The differential equation and the boudary condition can be both included in the following integral equation

$$S(\tau)=1+T\int_0^\tau d\tau_1 S(\tau_1)\Omega(q_0 p_0 \tau_1). \tag{11}$$

We are interested in the operator $S(1)$.

We would like to discuss now the time dependence of the operator Ω. The Hamiltonian never has dynamical time dependence since its total time derivative is equal to its partial time derivative. Therefore $H=H(q, p, \tau)$ depends on time τ only because of the explicit time dependence of the same. Besides Liouiville's operator may be written formally as

$$\Omega=H\,I \tag{12}$$

where I is the double operator defined as

$$I=\sum_i \frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q_i}-\frac{\overleftarrow{\partial}}{\partial q_i}\frac{\overrightarrow{\partial}}{\partial p_i}. \tag{13}$$

But I is invariant with respect to a canonical transformation. As a matter of fact, this is the actual content of the well-known invariance of the Poisson's brackets with respect to the above-mentioned transformations. Therefore I is invariant with respect to the time evolution of the system which is a canonical transformation. Consequently, Ω depends on τ only through the explicit time dependence of $H(\tau)$. Let us now pass to the interaction picture. We define $H_0\equiv H(0)$ as the unperturbed Hamiltonian, and $S_0(\tau)$ the evolution operator generated by H_0. We define the interaction picture evolution operator $S_1(\tau)$ by

$$A(q(\tau), p(\tau))=S_1(\tau)\,S_0(\tau)\,A(q_0, p_0)\,S_0^{-1}(\tau)\,S_1^{-1}(\tau). \tag{14}$$

The perturbation Hamiltonian is defined by

$$H_1(\tau)=H(\tau)-H_0, \tag{15}$$

Hamiltonian that depends on time explicitly. From it we can construct another Liouiville's differential operator that we will call $\Omega_1(q_0 p_0 \tau)$.

Entering into (5) with expression (14), we find out the integral equation satisfied by $S_1(\tau)$.

$$S_1(\tau)=1+T\int_0^\tau d\tau' S_1(\tau')\,\Omega_1[\tau'] \tag{16}$$

where

$$\Omega_1[\tau]=S_0(\tau)\,\Omega_1(q_0 p_0 \tau)\,S_0^{-1}(\tau) \tag{17}$$

is the perturbing Liouiville's operator in the interaction picture. Such operator depends on time τ on two accounts: Firstly because $H_1(\tau)$ has an explicit time dependence, secondly because the unperturbed evolution operators $S_0(\tau)$ acting on the canonical set of conjugate variables that enter in the definition of $\Omega_1(\tau)$ will induce the time dependence of the unperturbed motion in the same. Now, since the operator I of (13) is invariant under canonical transformations and remembering the symbolic expression (12), this second time dependence induced in $\Omega_1(\tau)$ corresponds to the time dependence induced in $H_1(\tau)=H_1(q, p, \tau)$ by $S_0(\tau)$; so, symbolically

$$\Omega_1[\tau]=H_1(q_0(\tau), p_0(\tau), \tau)\cdot I \tag{18}$$

where

$$q_{i0}(\tau)=S_0(\tau)\, q_{i0} S_0^{-1}(\tau)\ ;\ p_{i0}(\tau)=S_0(\tau)\, p_{i0} S_0^{-1}(\tau). \tag{19}$$

The slow time dependence of the perturbing Hamiltonian implies that $\Omega_1(q_0, p_0, \tau)$ does not contain the large parameter T, while $S_0(\tau)$ of (17) should more exactly be written, as

$$S_0(\tau)\rightarrow S_0(T\tau)=S_0(t) \tag{20}$$

when $t_0=0$ as we will suppose from now on.

§ 4. Periodic motion

We suppose that the unperturbed motion generated by H_0 is periodic. There are two kinds of periodic motions. Consider a system with a single degree of freedom. For such a system phase space is a two-dimensional plane. The first type, designated by the name "libration", occurs whenever both q and p are periodic functions of the time with the same frequency. Its orbit in phase space is closed. For the second type the coordinate q itself is not periodic, but is such that when q is increased by some value, the configuration of the system remains essentially unchanged. This motion will be referred to, simply, as rotation. The values of the position coordinate, that indeed in this type of periodicity is invariably an angle of rotation, are no longer bounded, but can increase indefinitely. In dealing with systems of more than one degree of freedom, the motion of the system is said to be periodic if the projection of the system point on each (q_i, p_i) plane is simply periodic in the sense defined for motion of only one degree of freedom. We suppose that the Hamilton-Jacobi equation is separable in at least one set of canonical variables. Then the projected motions are independent of each other, and their nature may be readily examined.

It is well known that, for periodic motions, when the value of the angle coordinate changes by unity the corresponding separation coordinate goes through a complete cycle. For the libration case this means return to its original value, while for the rotation type of periodicity the behaviour is more complicated. In general it is therefore possible to express a periodic separable coordinate as a Fourier series. The same may be said of a function of the separation coordinates such as, for instance, the perturbing Hamiltonian. We will limit ourselves to the cases when this Fourier expansion is possible.

We have to assume now that the necessary conditions are met so that the unperturbed periodic motion introduces a time dependence in $H_1(q_0(\tau), p_0(\tau), \tau)$ such that this function may be represented as a sum of simple harmonic motions involving variable coefficients that account for the explicit time dependence of H_1. So we may write

$$H_1[q_0(\tau), p_0(\tau), \tau] = \sum_j h_j(\tau)\, e^{i\omega_j \tau T}. \tag{21}$$

The coefficients $h_j(\tau)$ would not depend on time τ if the perturbing Hamiltonian did not have explicit time dependence. The constants ω_j represent the fundamental frequencies and all its harmonics of the unperturbed periodic motion.

§ 5. Approximate constancy of adiabatic invariants

Let us suppose that the functions $A(q, p)$ represent constants of the motion of the unperturbed motion. For them it holds

$$S_0(\tau)\, A(q_0 p_0)\, S_0^{-1}(\tau) = A(q_0 p_0) \tag{22}$$

and, therefore, its time evolution will be given by $S_1(\tau) A(q_0 p_0) S_1^{-1}(\tau)$. As usually done, the integral equation for $S_1(\tau)$ is solved by iteration. Thus we get the following expansion for $S_1(1)$.

$$S_1(1) = 1 + T\int_0^1 d\tau' \Omega_1[\tau'] + T^2 \int_0^1 d\tau' \int_0^{\tau'} d\tau'' \Omega_1[\tau''] \Omega_1[\tau']$$
$$+ \cdots T^n \int_0^1 d\tau' \int_0^{\tau'} d\tau'' \cdots \int_0^{\tau^{(n-1)}} d\tau^{(n)} \Omega_1[\tau^{(n)}] \Omega_1[\tau^{(n-1)}] \cdots \Omega_1[\tau'] + \cdots. \tag{23}$$

The general term in this series is equal to a multiple integral of a chronological ordered product of Liouiville's operators $\Omega_1[\tau]$. This fact is a well-known theorem in quantum mechanics, which has recently been extended to classical mechanics.[5)]

Therefore the m-th term of series (23) satisfies

$$T^n \int_0^1 d\tau' \int_0^{\tau'} d\tau'' \cdots \int_0^{\tau^{(n-1)}} d\tau^{(n-1)} \Omega_1[\tau^{(n)}] \cdots \Omega_1[\tau']$$
$$= \frac{T^n}{n!} \int_0^1 d\tau' \int_0^1 d\tau'' \cdots \int_0^1 d\tau^{(n)} \{\Omega_1[\tau'] \Omega_1[\tau''] \cdots \Omega_1[\tau^{(n)}]\}_- \tag{24}$$

where the symbol $\{\ \}_-$ means chronological ordering: It rearranges the product of time-labeled operators in the same order as the time sequence of their label, the latest one in time being last in the product, i.e. operators appear in the order, reading from left to right, of growing time values.

Let us now study the value of an integral whose integrand is a rapidly varying periodic operator as it is $\Omega_1[\tau]$ since in (21) we are going to consider T large. For instance we will study the value of

$$R_j(T) = \int_0^1 \Omega_j(\tau)\, e^{i\omega_j \tau T}\, d\tau \tag{25}$$

where $\Omega_j(\tau)$ is Liouiville's operator corresponding to $h_j(\tau)$. Integrating $R_j(T)$ m-times by parts and provided that $\Omega_j(\tau)$ and its first m derivatives are zero for $\tau=0$ and for $\tau=1$, we find

$$R_j(T) = \frac{(-1)^{m+1}}{(i\omega_j T)^{m+1}} \int_0^1 e^{i\omega_j \tau T} \frac{d^{m+1}\Omega_j(\tau)}{d\tau^{m+1}}\, d\tau, \tag{26}$$

integral that is quite small for large T. The operator $d^{m+1}\Omega_j(\tau)/d\tau^{m+1}$ is Liouiville's operator for $d^{m+1}h_j(\tau)/d\tau^{m+1}$. Let us now try to evaluate the action of $R_j(T)$ on the adiabatic invariants if the derivatives of order higher than m of $\Omega_j(\tau)$ are not zero at the initial and final times. We integrate again by parts and obtain

$$R_j(T) = \frac{(-1)^{m+1}}{(i\omega_j T)^{m+1}} \left\{ \left[\frac{e^{i\omega_j \tau T}}{i\omega_j T} \frac{d^{m+1}\Omega_j(\tau)}{d\tau^{m+1}} \right]_{\tau=0}^{\tau=1} - \frac{1}{i\omega_j T} \int_0^1 e^{i\omega_j \tau T} \frac{d^{m+2}\Omega_j}{d\tau^{m+2}}\, d\tau \right\}. \tag{27}$$

To find an upper limit of this expression for $T\to\infty$ we neglect the term containing the integral since by the same procedure of integration by parts we can show that such a term contains powers of $1/T$ higher than the other terms. Since the operator $d^{m+1}\Omega_j(\tau)/d\tau^{m+1}$ does not contain the parameter T, we can in general find an upper bound to the action of the same when it acts on a certain constant of the motion of the unperturbed system. Let us call M_1 and M_0 the upper bounds of $d^{m+1}\Omega_j(\tau)/d\tau^{m+1}$ at the instants $\tau=1$ and $\tau=0$ respectively. Then the action of $R_j(T)$ on such adiabatic invariant is bounded as follows,

$$\|R_j(T)\| < \frac{M_1 + M_0}{(\omega_j T)^{m+2}}. \tag{28}$$

The same can be said for any of the integrals whose sum is the right-hand side of the series (23). Applying identical procedure of partial integrations to each one of the n integrals of the general term (24) of (23), we will find that for large T such a term is of the order

$$\frac{1}{T^{(m+2).n}} \tag{29}$$

provided that $H_1(\tau)$ and its m first derivatives are zero at $\tau=0$ and $\tau=1$.

So we deduce that all the constants of the motion of the initial Hamiltonian H_0 are adiabatic invariants of order $m+1$ of the slowly time dependent Hamiltonian $H(\tau)$ provided that $H_1(\tau)$ and its m first derivatives are zero at the beginning and at the end of the interval $t_1-t_0=T$ which we suppose to be very large.

The precedent statement includes as a particular example the case of adiabatic invariance to all orders when $H_1(\tau)$ and all its derivatives are zero at the extremes of the long time interval. We may deduce for this case that the difference between $S_1(\tau)$ and 1 is smaller than any power of $1/T$. We know that in certain cases,[6] such a difference behaves as e^{-T}.

The above said conditions can be simplified if T is large but finite. Then the constants of the motion of the periodic motion are adiabatic invariants of order $m+1$ of the Hamiltonian $H(t)$ if this Hamiltonian and its m first derivatives are zero at the beginning and at the end of the time interval.

The present technique provides a method of evaluating the approximate constancy of the adiabatic invariants for large but finite T. Indeed for the constants of H_0 we have

$$A(q(1),p(1))-A(q(0),p(0))=\Delta A$$

$$=\Big[\sum_j \frac{T}{(i\omega_j T)^{m+1}}\int_0^1 e^{i\omega_j \tau T}\frac{d^{m+1}\Omega_j}{d\tau^{m+1}}d\tau,\ A(q(0),p(0))\Big] \tag{30}$$

since all the other terms of $S_1(1)$ contribute much less to this difference.

We would like to emphasize that this proof of the adiabatic invariance of the constants of the motion of the unperturbed Hamiltonian is not based on the smallness of the perturbation Hamiltonian, but rather we have required that the perturbation Hamiltonian evolving under the action of the unperturbed one be a purely oscillating function of time without non-oscillating components.

We are going to examine presently the adiabatic invariance of the constants of the motion of the Harmonic oscillator. We will see that the deduction of many well-known results implies a small modification of techniques developed so far.

We would like to remark here that the procedure of introducing the interaction picture for classical mechanics allows us not only to show when we have adiabatic invariants of m-th order, but also to evaluate the variation of such adiabatic invariants during the evolution of the system when T is large but finite, this last result being of great interest.

§ 6. A differential equation

In the same way as, from the differential equation of motion (4) of analytical dynamics, we can deduce an operational equation (7), for the adiabatic invariants whose time evolution is given by $S_1(\tau)$, we can deduce a differential equation

$$\frac{\delta A(\tau)}{\delta\tau}=T\sum_i \frac{\partial H_1[\tau]}{\partial p_i}\frac{\partial A(\tau)}{\partial q_i}-\frac{\partial H_1[\tau]}{\partial q_i}\frac{\partial A(\tau)}{\partial p_i} \tag{31}$$

corresponding to a Hamiltonian that is the perturbation Hamiltonian in interaction picture. With the help of this differential equation, stability conditions can be discussed.

§ 7. Application

We plan to study now the motion of a charged particle in a slowly time depending uniform magnetic field. In particular it is interesting to examine the approximate constancy of the magnetic moment of the charged particle.

The motion of such particle is essentially equivalent to that of a simple harmonic oscillator provided that we identify the Larmor frequency with twice the frequency of the oscillator.[7] The magnetic moment is proportional to the action variable of the oscillator.

Therefore in this paper we will limit ourselves to study the approximate constancy of the action variable for the simple harmonic oscillator.

The Hamiltonian for this problem is

$$H=\frac{p^2}{2m}+m\omega^2\frac{q^2}{2}. \tag{32}$$

Here ω is the frequency that we will suppose a slow function of time

$$\omega=\omega(\tau)=\omega\left(\frac{t}{T}\right). \tag{33}$$

The action integral J for a simple harmonic oscillator is equal to the energy divided by the frequency of the oscillation. Making use of the method described above, we can easily determine the change in J for a variation of ω.

The usefulness of performing a canonical transformation to the action-angle variables to study this kind of problems is well known. Therefore we shall make a canonical transformation from the set (q, p) of canonical conjugate variables to the new variables (θ, J) defined in such a way that they reduce to the ordinary action-angle variables when the Hamiltonian (32) does not contain explicit time dependence, i.e. when the oscillator has a constant frequency. As is usually done, we shall choose the generating function to depend on q, θ, and τ, and in such a way that it reduces for constant ω, to the generating function of the well-known[8] canonical transformation to the action-angle variables for the simple harmonic oscillator of constant frequency. Therefore the generating function is

$$G(q, \theta, \tau)=\frac{1}{2}m\omega(\tau)\,q^2\cot\theta. \tag{34}$$

which yields the following expressions for the momentum p and the action J,

$$p = m\omega q \cot\theta,$$
$$J = \frac{m\omega q^2}{2\sin^2\theta}, \tag{35}$$

and the new Hamiltonian written in terms of θ, J is

$$H' = H + \frac{d\tau}{dt}\frac{\partial G}{\partial \tau} = \omega J + \frac{1}{2T}\frac{\dot\omega}{\omega} J \sin 2\theta \tag{36}$$

where

$$\dot\omega \equiv \frac{d\omega}{d\tau} = T\frac{d\omega}{dt}. \tag{37}$$

The partition of H' into unperturbed term H_0' and perturbation in order to obtain the interaction picture is immediate,

$$H_0' = \omega J,$$
$$H_1' = \frac{1}{2T}\frac{\dot\omega}{\omega} J \sin 2\theta. \tag{38}$$

Let us examine if the conditions sufficient for the adiabatic invariance of J are fulfilled. The unperturbed equations of motion are

$$\frac{d\theta}{dt} = \omega(\tau) = \omega\left(\frac{t}{T}\right),$$
$$\frac{dJ}{dt} = 0, \tag{39}$$

whose solutions are

$$\theta_0(t) = \theta_0 + \int_0^t \omega\left(\frac{t'}{T}\right) dt' = \theta_0 + T\int_0^\tau \omega(\tau')\, d\tau',$$
$$J_0(t) = J_0 \tag{40}$$

where $\theta_0 = \theta_0(0)$, $J_0 = J_0(0)$ are the initial values of these variables. Indeed J is a constant of the motion of H_0'. The perturbing Hamiltonian $H_1' \equiv H_1'(\tau)$ in the interaction picture is

$$H_1'[\tau] = S_0(\tau)\, H_1'(\tau)\, S_0^{-1}(\tau) = \frac{1}{2T}\frac{\dot\omega}{\omega} J_0 \sin 2\left(\theta_0 + T\int_0^\tau \omega(\tau')\, d\tau'\right) \tag{41}$$

which for large T is a rapidly oscillating function of the general form (21) that fulfills all the sufficient conditions to assure the adiabatic invariance of J. In our case only one function $h_j(\tau)$ appears and it is $1/2T\cdot\dot\omega(\tau)/\omega(\tau)\cdot J_0$.

We would like to remark a peculiarity proper to the simple harmonic oscillator, that is, that $H_1'[\tau]$ is not quite a periodic function without constant Fourier term, but it is rapidly oscillating and such that $H_1'(\tau)$ of (38) is a small perturbation compared with H_0' property that is much stronger than those needed to prove the adiabatic invariance of the constants of the motion of H_0'. Indeed we have

$$\frac{H_1'}{H_0'} \simeq \frac{1}{2T} \frac{\dot{\omega}}{\omega^2} \tag{42}$$

that goes to zero for large T provided that ω never becomes zero, as is the case for a charged particle in a magnetic field. As a matter of fact the proofs presented so far to show the adiabatic invariance of the action integral of the harmonic oscillator were based on this peculiar property, that is, why they could not be applied to other mechanical systems.

From formula (30) we obtain the following value for the variation of the action integral in the interval $(0, T)$

$$\delta J \equiv J(1) - J(0) = -J_0 \int_0^1 \frac{\dot{\omega}(\tau)}{\omega(\tau)} \cos 2\left(\theta_0 + T \int_0^\tau \omega(\tau')\, d\tau'\right) d\tau \tag{43}$$

for whose deduction we have used

$$I = \frac{\overleftarrow{\partial}}{\partial J} \frac{\overrightarrow{\partial}}{\partial \theta} - \frac{\overleftarrow{\partial}}{\partial \theta} \frac{\overrightarrow{\partial}}{\partial J} \tag{44}$$

and the fact that

$$\frac{\partial \theta_0(t)}{\partial \theta_0(t_1)} = 1. \tag{45}$$

If $\dot{\omega}$ and its m first derivatives are zero at $\tau=0$ and $\tau=1$, i.e. if the first $m+1$ derivatives of $\omega(\tau)$ are zero at $\tau=0$ and $\tau=1$, the integral in (43) is of the order $1/T^{m+2}$. In such a case, therefore, the action integral is adiabatic invariant of order $m+2$. Observe that this is so even if $\omega(0) \neq \omega(1)$, i.e. in the case when the magnetic field in which the charged particle moves changes from a constant initial value B_0 to another constant final value B_1, provided that this change takes place in a very long time compared with the Larmor period.

Integrating (43) by parts we will obtain the corresponding value of δJ. Finally we should remark that sometimes it is convenient to evaluate the average of the resulting δJ over equally weighted values of θ_0; and to evaluate this average $\langle \delta J \rangle_{\theta_0}$ we may need second terms of the perturbation series (23) since this average of (43) is identically zero.

§ 8. Conclusions

We should like to call the reader's attention to a paper by S. Tamor[9] in which he extended the adiabatic invariance concept for a one-dimensional oscillator to the case where the frequency has one singularity for finite time. We have studied here only the case where the Hamiltonian does not possess such a singularity. However, the case of a singular Hamiltonian can be handled by the formalism here presented. The study of this problem and the new conclusions at which we arrive, will be done elsewhere.

The assumptions taken in this paper may be summarized stating that once the Hamiltonian is splitted into the unperturbed part and the perturbation, this last one considered as function of $q=q_0(t)$, $p=p_0(t)$ solutions of the unperturbed motion, should be a sum of rapidly oscillating terms when the time interval T becomes very large. Under these conditions the constants of the motion of the unperturbed Hamiltonian are adiabatic invariants of m-th order if the perturbing Hamiltonian and its $m-1$ time derivatives with respect to the explicit time dependence of the perturbation are zero at the initial and final time instant.

In order to compare this approach with the one by Kulsrud[3] we may say that ours is more general since it is based on an operational development of classical mechaics and on the introduction of the interaction picture. By our approach we can treat the adiabatic invariants of the harmonic oscillator as a particular example of our general theory as soon as we split the Hamiltonian into the unperturbed part and the perturbation as it is done in the application in which, precisely, we studied the harmonic oscillator. If this is done, the conclusions at which both approaches arrive coincide.

Finally we remark that, as it happened for the harmonic oscillator, the unperturbed Hamiltonian may be time depending also. Its constants of the motion are adiabatic invariants provided that the perturbation in the interaction picture fulfills the required conditions.

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